Simple Replication Krusell and Smith (1998)

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February 23, 2025

This time we try to replicate the result from Krusell and Smith (1998) without estimation part, and learn how to deal with aggregate variable moves in heterogeneous model. In the mean time, we try to learn the technique for doing interpolation of value function iteration.

1 Environment

- Unit measure of agents.
- Time period is one quarter.
- Preferences
$$\sum \beta \ln(c_t), \quad \beta = 0.99$$
- Production technology
$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$$
where $\alpha = 0.36$ and aggregate technology shocks $z_t \in \{z_g = 1.01, z_b = 0.99\}$ are drawn from a markov process with transition matrix probabilities $\pi_{ss'} = prob(z_{t+1} = s'|z_t = s)$
- Capital depreciates at rate $\delta = 0.025$
- Idiosyncratic employment opportunities ϵ_t. Since agents derive no utility from leisure and have 1 unit time, labor input $\epsilon_t = 1$ means the agent is employed with efficiency $\bar{e}$ receiving wage w_t per unit of labor efficient time and $\epsilon_t = 0$ means he is unemployed.
- Employment opportunities are correlated with the aggregate state of the economy. By virtue of the law of large numbers and iid employment opportunities across agents, the only source of aggregate uncertainty are aggregate shocks. More specially, the number of agents who are unemployed in the good state is always u_g and the number unemployed in the bad state is always u_b (i.e. when you control for z, individual shocks are uncorrelated).
- In terms of exogenous uncertainty, the markov transition matrix from state (z, ϵ) to (z', ϵ') is denoted $\pi_{zz',\epsilon\epsilon'}$: In sum, there are 12 parameters in the transition matrix (which takes into account the adding up constraint across each row) K-S use a set of restrictions/assumptions to pin down$\pi_{zz',\epsilon\epsilon'}$.
- Incomplete Asset Markets: Households rent their capital $k_t \geq 0$ to firms and receive rate of return r_t (strict borrowing constraint).
- The individual labor supply is $\tilde{l} = 0.3271$, implying each work work about 8 hours per day. (This is important)

2 Equilibrium

From the model , we can get the expression of w_t and r_t given K_t, L_t, z_t

$$w_t(K_t, L_t, z_t) = (1-\alpha)z_t(\frac{K_t}{L_t})^{\alpha}$$
$$r_t(K_t, L_t, z_t) = \alpha z_t(\frac{K_t}{L_t})^{\alpha-1}$$

And guess the form of aggregate capital law of motion

$$\log \bar{K}_{t+1} = a_0 + a_1 \log \bar{K}_t,\ \ z_t = z_g$$
$$\log \bar{K}_{t+1} = b_0 + b_1 \log \bar{K}_t,\ \ z_t = z_b$$

3 Computation

Aggregate Capital Law of Motion In order to replicate the model's result, we first select moment condition $M = 1$ with guess the aggregate capital law of motion has

$$\log \bar{K}_{t+1} = a_0 + a_1 \log \bar{K}_t,\ \ z_t = z_g$$
$$\log \bar{K}_{t+1} = b_0 + b_1 \log \bar{K}_t,\ \ z_t = z_b$$

Transition Matrix To calculate the transition matrix, since Krusell and Smith say (p. 877) that the average duration of both good and bad times is 8 quarters, this implies that the transition matrix for aggregate shocks is given by $\pi(z' = g|z = g) = \pi(z' = b|z = b) = 1/8$ and also the conditions for z ϵ we can get the transition matrix

Transition Matrix

	$g0$	$g1$	$b0$
$g0$	0.292	0.583	0.0938
$g1$	0.0243	0.851	0.00911
$b0$	0.0313	0.0938	0.525

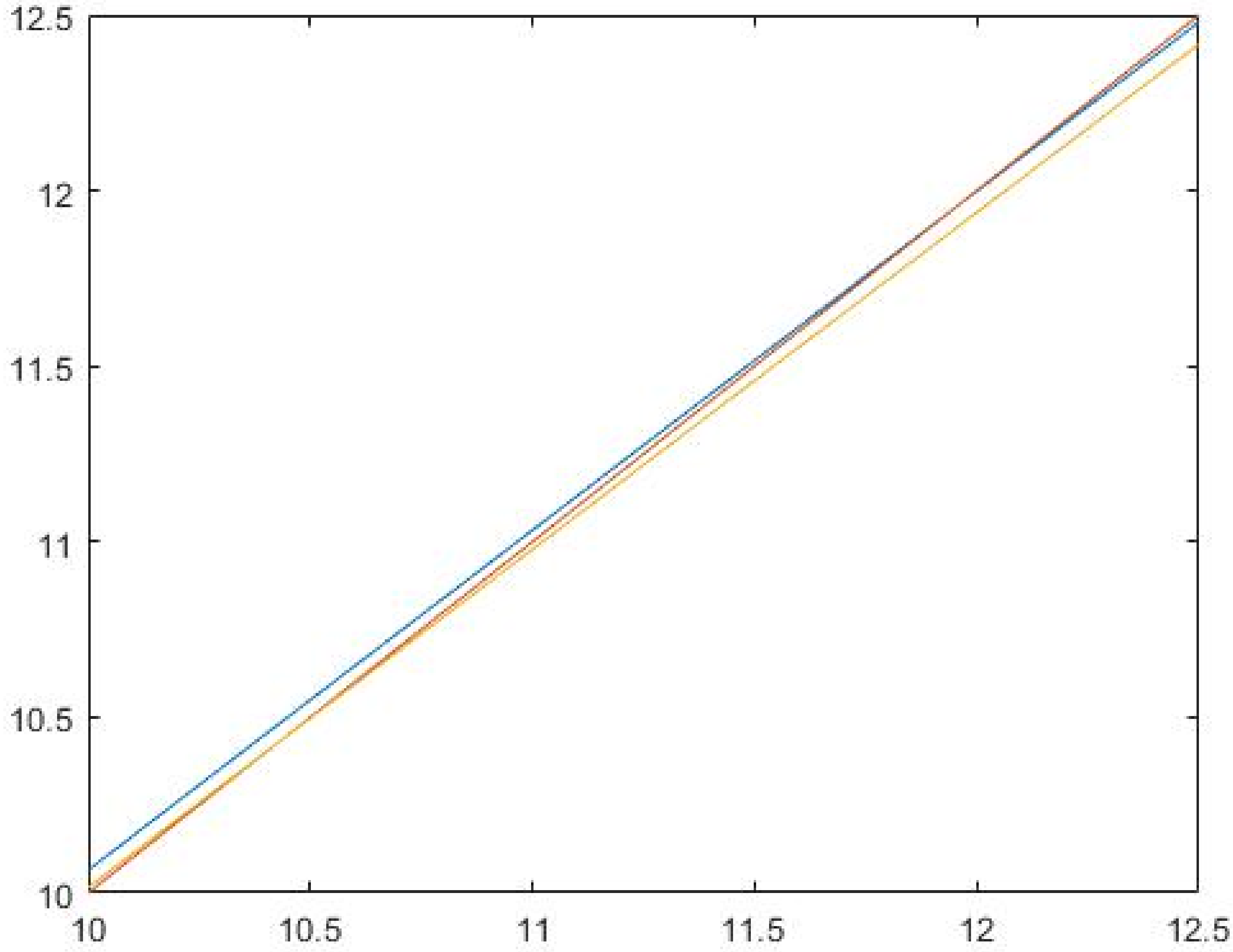
where the g, b represent the aggregate shock *good bad* $0, 1$ represent individual shock : unemployment and employment.

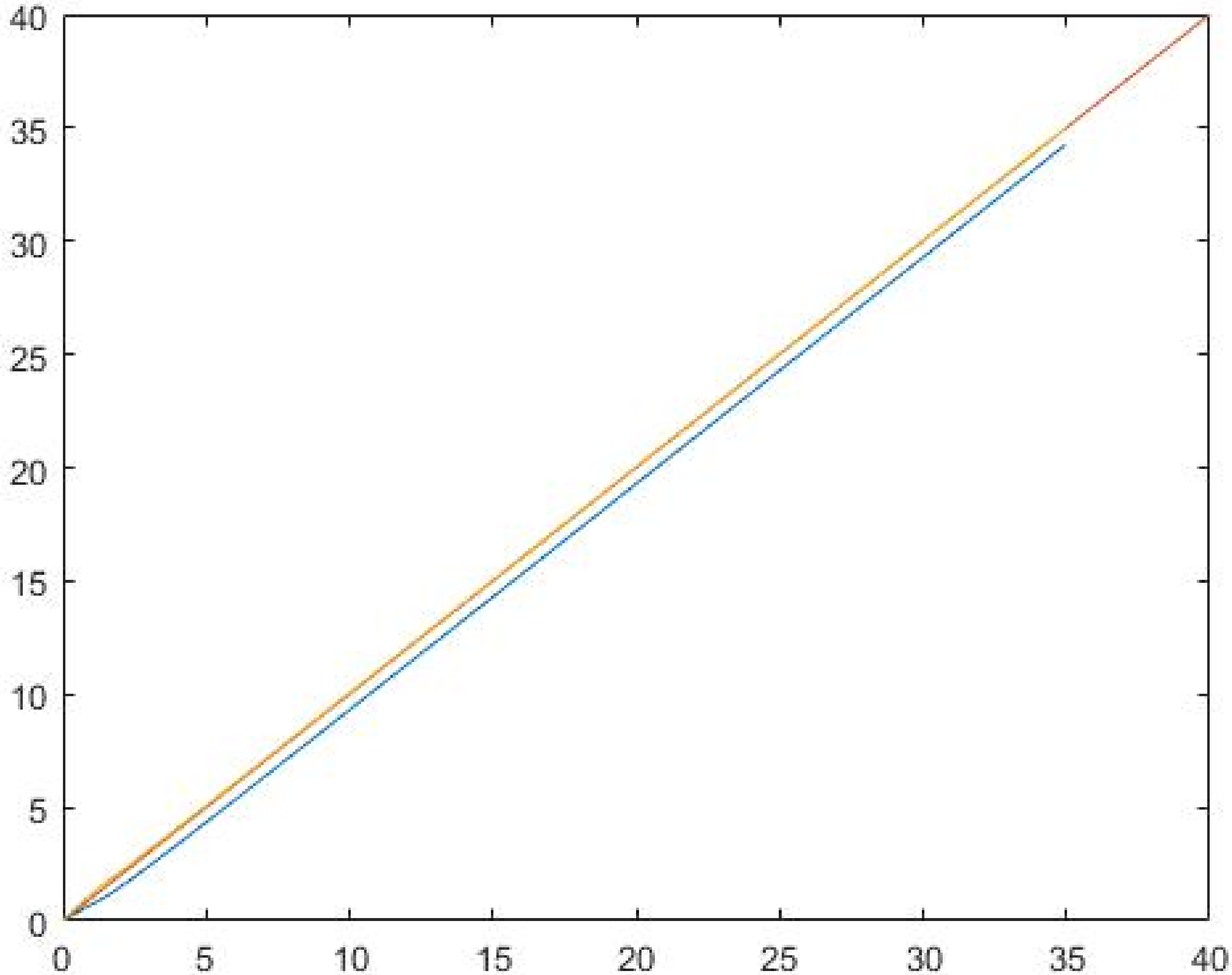
Value Function iteration and interpolation The objective of the numerical algorithm is to approximate the four functions $v(k, 1; \bar{k}, z_g)$, $v(k, 1; \bar{k}, z_b)$, $v(k, 0; \bar{k}, z_g)$, $v(k, 0; \bar{k}, z_b)$. The paper accomplish this task by approximating the values of each of these functions on a coarse grid of points in the$(k, \bar{k})$ plane and then using **cubic spline** and **polynomial interpolation** to calculate the values of these functions at points not on the grid.
For the aggregate capital $\bar{k}$, since there is generally not much curvature in the value function in the $\bar{k}$ direction, we use a small number of grid points in this direction and we use polynomial interpolation to compute the value function for values of $\bar{k}$ not on the grid. the grid point we select is 5 around the steady state of aggregate capital.
For the individual k, there is generally a fair amount of curvature in the value function, especially for values of k near the borrowing constraint (near 0) . In this direction, therefore, we use cubic spline interpolation, which fits a piecewise cubic function through the given function values, with one piece for each interval defined by the grid. We generally use 80 grid points in the k direction, with many half points near zero. Note that in order to precisely interpolate the k outside the grid point, we do the interpolation separately when k is around 0 .
And then we just use the approach described in the paper to combine the two interpolation schemes $(k, \bar{k})$. And get value function of $v(k, 1; \bar{k}, z_g)$, $v(k, 1; \bar{k}, z_b)$, $v(k, 0; \bar{k}, z_g)$, $v(k, 0; \bar{k}, z_b)$ in each loop. Because of the interpolation, the value function converges in a slower speed, thus when the policy function converges in a acceptable range, we can just finish the iteration. After finishing the VFI, we can use interpolation again to approximate the decision rules with more precise grids. For the purpose of approximating decision rules, we use 500 equally spaced points in the k direction and 50 equally spaced points in the k direction. Optimal decisions at points not on the grid are then computed using bilinear interpolation.

Simulation and Estimation We just follow the instruction from the paper to settle down the aggregate shock and random shock,for each period , using decision rule to settle down the next period k' and compute each period aggregate capital $\bar{k}$. When we get the data of $\bar{k_t}$, we can select $\bar{k}_{t+1}$ and $\bar{k}_t$ given $z_t = g$ into one sample and if $z_t = b$, place them into another. Run the two regressions to obtain the a and b coefficients.

4 Result

After the computation, we get the decision rule for individual capital k and aggregation capital $\bar{k}$





And the final result is

Regression Result			
	intercept	slope	R^2
good time	0.0893	0.9640	0.9997
bad time	0.0863	0.9632	0.9998

Therefore, we verify the result of the paper. Also we tried that if we set another parameter such as β the guess for a and b no longer holds.