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"CONVENTIONAL" MONETARY POLICY IN OLG MODELS: REVISITING THE ASSET-SUBSTITUTION CHANNEL*

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Conventional monetary policy involves actions by the monetary and fiscal authorities: the former sets a nominal interest rate and the latter sets lump-sum taxes to finance the implied flow of interest payments on government debt. We model such policy within an overlapping generations framework and show that absent any other frictions the magnitude of the nominal interest rate gives rise to asset substitution between government debt and either private debt or capital—substitution that has both real and nominal effects. Such substitution is not in standard New Keynesian models because their dynastic specification implies that government debt is not net wealth.

1. INTRODUCTION

A common view is that the interest rate set by the central bank, the nominal rate on nominal short-term government debt, affects the attractiveness of alternatives assets such as common stock and housing. For example, the long period of low interest rates that began with the 2008 financial crisis is often claimed to have led to a flight toward real and risky assets, and the tightening of monetary policy in 2022 is viewed as having effects in the opposite direction. Such asset-substitution effects are missing in most, if not all, standard New Keynesian (NK) models and real business cycle (RBC) models because they are dynastic models in which government debt is not net wealth. Although it is well known that government debt is net wealth in overlapping generations (OLGs) models, no one has drawn attention to what such models imply for the effects of conventional monetary policy.

The failure to do so is somewhat surprising because OLG models are widely used for a variety of purposes. It has become common to study various aspects of taxation and insurance within life-cycle models either without bequest motives or with *warm-glow* bequest motives (Braun et al., 2019; Heathcote et al., 2020). Moreover, in several recent papers, aspects of macroeconomic policy are studied in such models: see, for example, Kaplan et al. (2018), Caballero and Farhi (2017), and Blanchard (2019). Although it is well known that Ricardian equivalence fails in such models, no one has drawn attention to an obvious implication of such models for *conventional* monetary policy: even in the absence of any frictions, conventional

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monetary policy has real consequences in such models because it determines the return on government debt and, thereby, the composition of the public's portfolio between government debt and other assets. We call attention to that implication here and call it *revisiting* the asset-substitution channel because it harkens back to Tobin's views about the effects of monetary policy (Tobin, 1969).

For us, and we think that this is standard, *conventional* monetary policy is defined as follows: the monetary authority sets a nominal interest rate (a path or rule for it) and the fiscal authority cooperates by setting lump-sum taxes to cover the implied flow of interest payments on (net) government debt (Woodford, 2010). Part of what is new here is our simple conception of conventional monetary policy. We assume that there is a positive and constant nominal stock of government debt and that the monetary authority sets the path of the nominal interest rate on that debt with interest payments financed by lump-sum taxes. We treat the debt as perpetuities with a time-varying nominal dividend, a treatment that is equivalent to having one-period debt that is rolled over period after period. In accord with the above definition of conventional monetary policy, lump-sum taxes are levied only to cover interest payments; they do not finance redemption of the debt. The amount of nominal government debt does not matter, provided that it is positive. Therefore, we normalize it to be unity per capita. We could, we think, also allow it to be negative, but we do not pursue that here. Only zero would be problematic.

Such debt differs from *money* in two respects: it has a nominal yield financed by lump-sum taxes and in order for it to be valuable it must be part of an optimal private-sector portfolio based solely on its rate-of-return distribution. If there is no uncertainty, then it and other assets must have the same real rate of return if they are held. The debt resembles money in that there is always an equilibrium in which the debt is worthless. We, of course, focus on equilibria in which it is valuable. Also, we omit *cash*, a nominal asset with a zero nominal return, a feature we share with most NK and RBC models.[1] Because our main purpose is to revive a debate about a channel for the effects of monetary policy, we describe that channel in four simple OLG models.

Throughout, we abstract from growth. Three of our models are deterministic. In them, it is well known that a competitive equilibrium, our concept of equilibrium, is Pareto-efficient if there is convergence to a positive real interest rate. All our equilibria, by construction, satisfy that condition. It follows that the role of monetary policy in those models is distributional. The welfare economics of stochastic OLG models is not straightforward. Therefore, for our stochastic model, we have only descriptive results.

The first model is a deterministic pure-exchange setting in which government debt competes with private borrowing. The second and third models are versions of Diamond (1965) in which saving is exogenous so that any effect of conventional monetary policy comes from asset substitution. The fourth model is a version of Blanchard (1985), a continuous-time model. We use it, in part, to study a more plausible relationship between investment and the capital stock than is possible in Diamond (1965).

As is well known, Diamond (1965), Blanchard (1985), and Blanchard (2019) study the role of real debt in OLG models. We, in effect, show that the quantity of real debt in their models can come about from a given and fixed amount of nominal debt and different settings of the nominal interest rate under conventional monetary policy. Finally, because our proofs are standard, we relegate them to an appendix.

[1] A version of the so-called Tobin effect shows up in models with cash and capital when the return on cash is varied by way of different inflation rates produced by different rates of money creation achieved through lump-sum taxes and transfers. See Bhattacharya et al. (2009) and Aruoba et al. (2011). Such policies do not, however, resemble conventional monetary policy.

2. A PURE-EXCHANGE OLG MODEL OF TWO-DATE LIVED PEOPLE

We let the first date be $t = 1$ and let generations be labeled by when they first appear. There is one good per date, N people in a generation, and the setting is stationary in the following sense. Person n in generation $t \geq 1$ has a utility function $u^n(c_t^{n,t}, c_{t+1}^{n,t})$ and an endowment $(w_t^{n,t}, w_{t+1}^{n,t}) = (a^n, b^n) \in R^2_{++}$, where the subscript denotes the date and the superscript identifies the person and where c denotes consumption. We assume that u^n is strictly increasing, strictly concave, continuously differentiable, satisfies Inada-type conditions that insure that consumption choices are both positive whenever the after-tax budget set is not empty, and is such that both goods are *normal* goods. As regards members of generation 0, each owns some amount of government debt, which in total is N, and some positive amount of the date-1 good, which in total is $\sum_{n=1}^N b^n$, and wants to consume as much as possible. An allocation is feasible if $\sum_{n=1}^N (c_t^{n,t} + c_t^{n,t-1}) \leq \sum_{n=1}^N (a^n + b^n)$.

We let p_t denote the unit price of debt at date-t in units of the date-t consumption good. We assume that the interest rate on government debt, i_t, is in units of the debt. People when old at t are subject to real lump-sum taxes that total $Np_t i_t$ over the members of generation $t-1$. We let r_{t+1} denote the one-period real interest rate earned on saving at date-t. We also let Np_t be the total ex dividend value of the government debt at date-t.

The choice problem of person-n of a generation $t \geq 1$ is as follows:

Problem 1. *Choose* $(c_t^{n,t}, c_{t+1}^{n,t}) \in R^2_+$ *to maximize* $u^n(c_t^{n,t}, c_{t+1}^{n,t})$ *subject to*

$$c_t^{n,t} + c_{t+1}^{n,t}/(1 + r_{t+1}) \leq a^n + (b^n - \tau_{t+1}^n)/(1 + r_{t+1}), \tag{1}$$

where $\tau_{t+1}^n \geq 0$ *is the value of the lump-sum tax payable when old.*

An equilibrium in which government debt is worthless is defined as follows:

DEFINITION 1. An equilibrium with $p_t = 0$ for all $t \geq 1$ and $\tau_{t+1}^n = 0$ for all n and $t \geq 0$ is a sequence $\{r_{t+1}\}_{t=0}^\infty$ and a feasible allocation that solves Problem 1 and satisfies $\sum_{n=1}^N c_t^{n,t} = \sum_{n=1}^N a^n$.

An equilibrium in which government debt is valuable is defined as follows:

DEFINITION 2. Let $\{i_{t+1}\}_{t=1}^\infty$ be given. An equilibrium with $p_t > 0$ for all $t \geq 1$ is a sequence $\{p_t, r_{t+1}\}_{t=1}^\infty$, a feasible allocation, and $\{\tau_{t+1}^n\}_{t=0}^\infty$ for each n such that (i) the right-hand side of (1) is positive for all n and $t \geq 1$; (ii) Problem 1 is solved; (iii) $(1 + i_{t+1})p_{t+1}/p_t = (1 + r_{t+1})$ for all $t \geq 1$; and (iv) $\sum_{n=1}^N \tau_{t+1}^n = Np_{t+1}i_{t+1}$ for $t \geq 0$.

An equilibrium with $p_t = 0$ and no taxes is an equilibrium in which there is no trade between generations, the last condition in Definition 1. We have nothing new to say about such equilibria. In particular, as is well known, nothing insures that such equilibria are efficient. However, because we want to emphasize that our analysis of monetary policy does not depend on inefficiency, we limit consideration to Definition 1 steady states that are efficient. We therefore make the following assumption:

ASSUMPTION 1. *Let* $p_t = 0$ *for all* $t \geq 1$ *and* $\tau_{t+1}^n = 0$ *for all* n *and* $t \geq 0$. *The preferences and endowments are such that (i) there exists a Definition 1 equilibrium with* $r_{t+1} = r^* > 0$ *and (ii) if* $r > r^*$, *then the solution to Problem 1 satisfies* $\sum_{n=1}^N c_t^{n,t} < \sum_{n=1}^N a^n$.

This assumption is obviously nonvacuous. We can then state and prove the following two propositions:

PROPOSITION 1. *Let $i_{t+1} = i > r^*$ for all $t \geq 1$ (see Assumption 1). There exists a Definition 2 equilibrium with $r_{t+1} = i$ and $p_t = p > 0$ for all $t \geq 1$.*

Now we describe the effects of a one-period deviation from such a steady state.

PROPOSITION 2. *Let $i > r^*$ and let $\{i_{t+1}\}_{t=1}^{\infty}$ be such that $i_{t+1} = i$ for $t \geq 2$, and let $p > 0$ be the Proposition 1 constant per capita value of the debt determined in a Proposition 1 equilibrium. Then, for any i_2, there exists an equilibrium with $p_t > 0$ for all t in which the real interest rate at date-1 is higher than i if $i_2 > i$ and is lower than i if $i_2 < i$.*

Notice that under Assumption 1, any Proposition 1 equilibrium or any Proposition 2 equilibrium is Pareto-efficient. Because the real return sequences differ among these equilibria as do the lump-sum taxes, those equilbria have different allocations, and hence, are noncomparable. In general, the initial old, the members of generation 0, fare differently among those equilibria. And, in general, borrowers and lenders also fare differently among them.

3. CAPITAL IN AN OLG MODEL OF TWO-DATE LIVED PEOPLE

Here, there is no diversity within a generation. The term *per capita* means division by the size of a generation. Each person supplies one unit of labor when young and nothing when old and maximizes the expected utility of consumption when old. Hence, per capita saving is the wage. The technology is a standard constant-return-to-scale technology and labor and capital earn their respective marginal products. We let k_t denote the per capita stock of capital entering date-t, the only initial condition, and let $zf(k_t)$ be per capita output at date-t, where $z > 0$ and f is twice differentiable, strictly increasing, and strictly concave. We also assume that capital fully depreciates in a period. Because saving is unaffected by the return on saving, any real response to monetary policy arises from asset substitution. Here, each young person's portfolio, which consists of some government debt and some capital, is determined by the condition that rates of return on the two are equalized. Under the assumption that capital and labor earn their marginal products, the date-t per capita wage is

$$w_t = z(f(k_t) - k_t f'(k_t)), \tag{2}$$

which is also per capita saving. That saving is split between an amount that will be k_{t+1}, the gross yield on which is $zf'(k_{t+1})$, and an amount that is the real value of the per capita debt of the government at t. As above, a lump-sum tax payable when old finances interest on the debt.

Therefore, we have the following definition of equilibrium with valuable debt:

DEFINITION 3. Given k_1 and a sequence for i_{t+1} for $t \geq 1$, a sequence for (k_{t+1}, p_t) for $t \geq 1$ with $p_t > 0$ for all t is an equilibrium if

$$w_t = k_{t+1} + p_t, \tag{3}$$

and

$$zf'(k_{t+1}) = p_{t+1}(1 + i_{t+1})/p_t. \tag{4}$$

We start by defining and describing *steady states* with valuable government debt. Given $i_{t+1} = i$ for $t \geq 1$, a steady state with $p > 0$ is (k, p) such that $(k_{t+1}, p_t) = (k, p)$ is an equilibrium for the initial condition $k_1 = k$. By (3), such a steady state exists *iff* $w > k$. By (2), that inequality is equivalent to

$$zf(k)/k > 1 + zf'(k). \tag{5}$$

We impose (5) by way of the following assumption:

ASSUMPTION 2. *The function $zf(\cdot)$ is such that there exists $k^* > 0$ so that if $k \in (0, k^*)$, then (5) is satisfied.*

It follows from this assumption that if $i^* = zf'(k^*) - 1$, then there is a unique steady state with valued government debt for any $i > i^*$. Notice that if $f(k) = k^\alpha$ with $\alpha \in (0, 1)$, then (5) reduces to $z(1-\alpha) > k^{1-\alpha}$, so Assumption 1 holds with $\ln k^* = \ln(z(1-\alpha))/(1-\alpha)$. Also, if f is CES with elasticity of substitution that exceeds unity, then Assumption 2 holds. It does not hold if the elasticity of substitution is less than unity.

It will be convenient to work with a second-order difference equation in k_t. Recall that w_t is the function of k_t given in (2). We use (3) to express p_t in terms of k_t and k_{t+1} and p_{t+1} in terms of k_{t+1} and k_{t+2} and substitute those expressions into (4), and we obtain

$$k_{t+2} = w_{t+1} - zf'(k_{t+1})(w_t - k_{t+1})/(1+i). \tag{6}$$

There is a single initial condition and one-side condition: $w_t > k_{t+1}$ (see (5)).

Let us start from the steady state for $t = 1$. We are interested in the effect of a one-date deviation of i_2 from its steady-state magnitude, a kind of impulse-response function. We describe it in two steps. We first show that the above steady state is locally saddle-path stable. Let k be the unique steady state associated with i. Let k_2 be an arbitrary initial condition at date-2 in the appropriate neighborhood of k. Then, associated with each such k_2, there is an associated equilibrium that also gives us an associated p_2. In the next step, we can associate with each such (k_2, p_2) an i_2 that satisfies the equilibrium conditions, (3) and (4), for $t = 1$. In particular, because w_1 is implied by k , k_2 implies p_1 by way of (3). Then, we can use (4) to find the implied i_2. If this mapping from k_2 to i_2 is monotone, then it can be inverted. If so, then we have the impulse response function. The part from $t = 1$ to $t = 2$ is given by the inverse of the mapping from k_2 to i_2; the rest is given by the local stability result.

PROPOSITION 3. *Assume that $i > zf'(k^*) - 1$ (see Assumption 2).*

1. *The unique steady state with valued government debt is locally saddle-path stable.*
2. *Starting from the unique steady state with valued government debt, a one-period increase in the nominal interest reduces the next period's capital stock.*

For a numerical example, we display the impulse response functions in the Online Appendix.

4. RISKY CAPITAL IN AN OLG MODEL OF TWO-DATE LIVED PEOPLE

In essentially all NK and RBC models, there are TFP shocks that make capital a risky asset. Here, we adopt the same model as in the previous section, except that we add such shocks by making z in the previous model random. In the presence of such shocks, there is a portfolio decision for each young person at t. We assume that each chooses a portfolio of capital and government debt to maximize the expected value of $u(c_{t+1})$, where, as we now assume, u is strictly increasing and strictly concave.

The date-t realization of z is denoted as z_t, which is assumed to follow a first-order Markov process on a finite support $\mathbb{Z}$. Then the state of the economy at t is (k_t, z_t). In general, we formulate policy, the choice of the nominal interest rate, as the choice of a function of the state;

say, $i_{t+1} = g(k_t, z_t)$. As this suggests, we assume the following timing. Entering date-t, k_t is inherited from the past. Then z_t is realized. Then i_{t+1} is determined. Finally, the young choose their portfolio.

We do not see a simple way to describe the kind of g functions consistent with an equilibrium with valued government debt. One way to find *a* class of such functions is to use a version of *backsolving*. In particular, we find the g function that supports an equilibrium that has a constant ratio of the value-of-debt to the wage. Call that ratio γ. Then, we would have

$$k_{t+1} = (1-\gamma)w_t \text{ and } p_t = \gamma w_t. \tag{7}$$

This is simple because for a given γ, we have a closed-form expression for how the state, (k_t, z_t), evolves and how p_t evolves, given that w_t depends on (k_t, z_t) in a known way.

All that remains is to find the process for i_{t+1} that makes γ an optimal portfolio choice. Here is the portfolio choice problem.

Problem 2. *Choose* (k_{t+1}, b_{t+1}) *to maximize* $\mathbb{E}u(c_{t+1})$ *subject to*

$$c_{t+1} = (1+i_{t+1})p_{t+1}b_{t+1} + z_{t+1}k_{t+1}f'(\bar{k}_{t+1}) - i_{t+1}p_{t+1}\bar{b}_{t+1},$$

and

$$w(k_t, z_t) = k_{t+1} + p_t b_{t+1},$$

where $p_{t+1} = \gamma z_{t+1}(f(\bar{k}_{t+1}) - f'(\bar{k}_{t+1})\bar{k}_{t+1})/\bar{b}_{t+1}$ *and* $\mathbb{E}$ *denotes expectation taken with respect to the distribution of the random variable* z_{t+1}.

In this choice problem, the individual takes aggregate capital stock $\bar{k}_{t+1}$ and aggregate debt stock $\bar{b}_{t+1}$ as given. The associated g comes directly from the first-order condition for the above choice problem after equating $(\bar{k}_{t+1}, \bar{b}_{t+1})$ to (k_{t+1}, b_{t+1}).

Now, in place of Assumption 2, we make the following assumption about the wage function, $w(z,k) \equiv z(f(k) - f'(k)k)$:

ASSUMPTION 3. *The function* $w(z,k)$ *is increasing and concave in* k, *in addition,* $\lim_{k\to 0^+} \frac{w(z,k)}{k} = +\infty$, *and* $\lim_{k\to +\infty} \frac{\partial w(z,k)}{\partial k} < 1$.[2]

Then we have the following result:

PROPOSITION 4. *Assume (7) and fix* $\gamma \in (0,1)$. *There exists a unique equilibrium with an associated ergodic set for the state* (k_t, z_t) *that is supported by the following rule for the nominal interest rate:*

$$1 + i_{t+1} = \frac{f'((1-\gamma)w_t)w_t}{f((1-\gamma)w_t) - f'((1-\gamma)w_t)(1-\gamma)w_t}. \tag{8}$$

Because w_t depends on the state at t, so does i_{t+1}. We suspect that this simple expression for i_{t+1} comes about because (7) implies that for any given i_{t+1}, the gross returns on capital and government debt are perfectly correlated.

We compute average impulse response functions for this model following the procedure described in Koop et al. (1996). We assume that $\mathbb{Z} = \{9, 11\}$ and that $\Pr(z_{t+1}|z_t) \equiv 1/2$, a uniform *i.i.d.* process for the shock. We also let $f(k) = k^{1/3}$ and $u(c) = \log(c)$. We simulate

[2] It is straightforward to verify that both the Cobb–Douglas production function and the CES production function with elasticity of substitution greater than 1 satisfy these conditions for all $z > 0$.

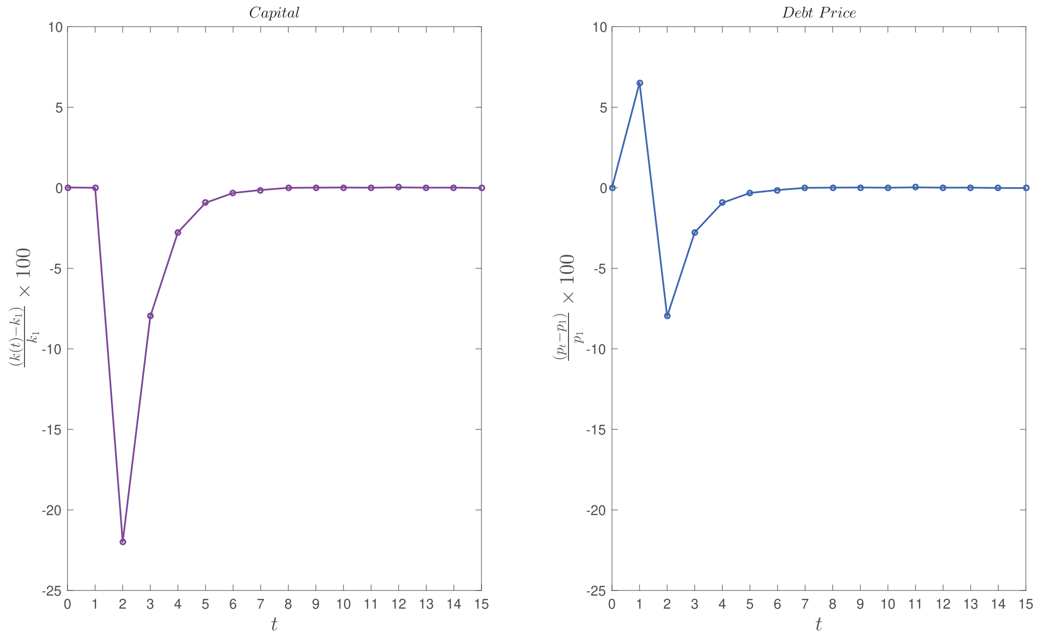

FIGURE 1

IMPULSE RESPONSE FUNCTIONS OF k_t AND p_t

20,000 economies for 265 periods with $t \in \{-249, -248, \ldots, -1, 0, 1, \ldots, 15\}$. At each period t, given z_t and k_t, we first calculate the wage $w_t = (1-\alpha)z_t k_t^{\alpha}$, and then, update $k_{t+1} = (1-\gamma_t)w_t$ and $p_t = \gamma_t w_t$. The initial capital stock $k_{-249} = 1.8764$ and $\gamma_t = 0.7718$ for $t \neq 1$, which are chosen to maintain the consistency with other impulse response functions in the Online Appendix.[3]

For $t = 1$, we impose the one-date shock $\gamma_1 = 0.8220$ that corresponds to the value of i_2 in the deterministic case in the Online Appendix. We drop the first 249 periods, which are used to approximate an ergodic distribution. We then calculate the mean of k_t and p_t across economies, respectively. We report these "average" impulse responses starting at 0 and relative to their averages for $\gamma = 0.7718$. Figure 1 displays the impulse response functions for k_t and p_t.

These average responses are very similar to what we found for the deterministic example of the second model (see Online Appendix).

5. AN OLG MODEL WITH LONG-LIVED PEOPLE AND CAPITAL

In this section, we use the framework of Blanchard (1985) because it allows us to choose both a more realistic depreciation rate of capital and a more realistic mortality rate. Instead of using the model to study different paths of real debt as in Blanchard (1985), we assume that the government debt is a fixed nominal quantity and that the nominal interest rate path on it is determined by monetary policy. As may be obvious, a path of real debt in Blanchard (1985) can be replicated by a path of the nominal interest rate in our model.

[3] We treat a period as 20 years. We assume a steady-state real interest rate of 4% per year (i.e., the average real interest rate in the United States from 1970 to 2018) and a deviation that raises i by 1.3% per year (i.e., one standard deviation of cyclical part of real interest rate). That is, $i = (1+0.04)^{20} - 1 = 1.1911$, which implies that $\gamma = 0.7718$, and $i_2 = (1.04 + 0.013)^{20} - 1 = 1.8091$, which implies that $\gamma_1 = 0.8220$. Also, the level of capital (k) at the steady state is 1.8764.

As in Blanchard (1985), at time t, an agent born at time s has consumption $c(s,t)$, owns capital stock $k(s,t)$, and holds nominal debt $b(s,t)$. The agent also provides one unit of labor supply to earn the labor income $w(t)$. The capital stock, which depreciates at rate δ, generates the marginal return $r(t)$ at time t. Given the production function $zf(\cdot)$ as in the previous models, we get $r(t) = zf'(K(t))$, $w(t) = zf(K(t)) - zf'(K(t))K(t)$, where $K(t)$ is the aggregate capital at time t that will be defined below. The price of the nominal debt at time t is $p(t)$ and interest rate on the nominal debt is $i(t)$. Given the constant probability of death ρ, the time discount rate θ, and logarithmic instantaneous utility, the agent maximizes:

$$\max_{\{c(s,t),b(s,t)\}_{v=t}^{\infty}} \int_t^{\infty} \exp\{-(\rho+\theta)(v-t)\} \log c(s,v)\mathbf{d}v. \tag{9}$$

At time t, each agent earns income $w(t)$ from labor, the effective capital return $(r(t) - \delta + \rho)k(s,t)$ from capital, and the effective debt return $(i(t) + \rho)p(t)b(t)$. They also pay the lump-sum tax $\tau(t)$. Thus, the dynamic budget constraint is

$$\begin{aligned} &c(s,t) + \dot{k}(s,t) + p(s,t)\dot{b}(s,t) \\ &\leq (r(t) - \delta + \rho)k(s,t) + w(t) + (i(t) + \rho)p(t)b(s,t) - \tau(t), \end{aligned} \tag{10}$$

and the transversality condition is

$$\lim_{v\to\infty} (k(s,v) + p(s,v)b(s,v)) \exp\left\{-\int_t^v (r(\mu) - \delta + \rho)\mathbf{d}\mu\right\} = 0. \tag{11}$$

The agent maximizes (9) subject to (10) and (11). It follows that the first-order condition is

$$\frac{1}{(\rho+\theta)} c(s,t) = a(s,t) + h(s,t), \tag{12}$$

where $a(s,t) = k(s,t) + p(s,t)b(s,t)$ and $h(s,t)$, is the discounted total labor income net of lump-sum tax at time t; namely,

$$h(s,t) = \int_t^{\infty} \left[(w(v) - \tau(v)) \exp\left\{-\int_t^v (r(\mu) - \delta + \rho)\mathbf{d}\mu\right\}\right]\mathbf{d}v.$$

The condition for the household to hold positive amounts of both capital and debt is

$$r(v) - \delta = i(v) + \frac{\dot{p}(v)}{p(v)}. \tag{13}$$

Before we define an equilibrium of this economy, for any function $x(s,t)$, we let

$$X(t) = \int_{-\infty}^{t} \rho x(s,t) \exp\{\rho(s-t)\}\mathbf{d}s.$$

Using this notation, the aggregated budget constraint can be written as

$$C(t) + \dot{A}(t) = (r(t) - \delta)A(t) + w(t) - \tau(t).$$

Given that the aggregate supply of government debt is normalized to be unity per capita and that $\tau(t) = i(t)p(t)$, we have the following definition of an equilibrium:

DEFINITION 4. Taking as given the interest rate path $i(t)$, an equilibrium is a solution to the following dynamic system:

$$\dot{C}(t) = (r(t) - \delta - \theta)C(t) - \rho(\rho + \theta)(K(t) + p(t)); \tag{14}$$

$$\dot{K}(t) = zf(K(t)) - \delta K(t) - C(t); \tag{15}$$

$$\frac{\dot{p}(t)}{p(t)} = r(t) - \delta - i(t), \tag{16}$$

with boundary conditions $K(0) = K_0$ and the transversality condition

$$\lim_{v\to\infty}(K(v) + p(v))\exp\left\{-\int_t^v (r(\mu) - \delta + \rho)\mathbf{d}\mu\right\} = 0. \tag{17}$$

Equation (14) is obtained as follows: First, find the aggregated law of motion for total asset $\dot{A}(t)$, and for the aggregate discounted posttax income $\dot{H}(t)$. Then integrate Equation (12) over different generations to derive the aggregated first-order condition. Differentiating this aggregated first-order condition with respect to t and eliminating $\dot{A}(t)$ and $\dot{H}(t)$, we obtain (14). Equation (15) is the goods market clearing condition, and Equation (16) is the no-arbitrage condition.

We first consider steady-state equilibrium with valued government debt.

PROPOSITION 5. *There exists a unique $i^* \in (\theta, \theta + \rho)$ such that if $i \in (i^*, \rho + \theta]$, then there is a unique steady state with $p^* > 0$.*

To study the effects of different paths for $i(t)$ from a given initial condition, we assume that the policymaker deviates from the steady-state interest rate i^* to $i(0)$, where $i(0) = i^* + e(0)$, where the initial shock to $i(0)$ decays exponentially at rate $-\eta$. The equilibrium path after this unexpected shock is given by the solution to (14)–(16) and

$$\dot{e}(t) = -\eta \cdot e(t), \tag{18}$$

with initial conditions $e(0) = e_0$, $K(0) = K^*$ and the transversality condition (17), where $i(t) = i^* + e(t)$.

To investigate the local dynamics and stability of this nonlinear system, we apply the Hartman–Grobman theorem (Arrowsmith and Place, 1992). The Jacobian matrix evaluated at the steady state is

$$J = \begin{bmatrix} zf'(K^*) - \delta - \theta & C^* zf''(K^*) - \rho(\rho+\theta) & -\rho(\rho+\theta) & 0 \\ -1 & zf'(K^*) - \delta & 0 & 0 \\ 0 & zf''(K^*)p^* & 0 & -p^* \\ 0 & 0 & 0 & -\eta \end{bmatrix}.$$

PROPOSITION 6. *Suppose $K(0) = K^*$.*

1. *The characteristic equation of J has two roots with strictly negative real parts and two with strictly positive real parts. Therefore, by the Hartman–Grobman theorem, the steady state in Proposition 5 is saddle-path stable.*
2. *We have*

$$\lim_{t\to 0^+}\frac{\partial \dot{K}(t)}{\partial e_0} < 0.$$

The second result confirms the Tobin effect. For a numerical example, we display the impulse response functions in the Online Appendix.

6. CONCLUDING REMARKS

We have pointed out what seems to be an overlooked consequence of using OLG models to study conventional monetary policy; namely, that such policy gives rise to asset substitution between government debt and other assets. And, although we have illustrated the asset-substitution role of conventional monetary policy in models without any frictions, it seems very unlikely that such a role would disappear in more general OLG models. Given the potential importance of that consequence, it seems that an old issue should be revisited: do we favor the dynastic setting, which currently is the dominant model for studying monetary policy, or some sort of OLG specification? That question used to be actively debated (Abel and Bernheim, 1991; Barro, 1974; Bernheim and Bagwell, 1988). Now, it is barely mentioned. That seems undesirable. The choice between the two models has consequences for many of the questions analyzed with either model and certainly for the role of conventional monetary policy.

APPENDIX A: PROOFS

The Appendix contains the proofs.

Proof of Proposition 1. PROOF. We use the intermediate value theorem. The main task is to construct a tax scheme. Given $r = i$, we know the pretax wealth of each person. Hence, we can make the tax on each person a share of their pretax wealth. Let $\sum_{n=1}^{N} a^n = A$ and $\sum_{n=1}^{N} b^n = B$. Then, let $\theta^n = [a^n + b^n/(1+i)]/[A + B/(1+i)]$. It follows that $\theta^n \in (0,1)$ and that $\sum_{n=1}^{N} \theta^n = 1$. Under this tax scheme, the budget set of person n is

$$c_t^{n,t} + \frac{c_{t+1}^{n,t}}{1+i} \le \left(a^n + \frac{b^n}{1+i}\right)\left[1 - \frac{Npi}{A(1+i)+B}\right]. \tag{A.1}$$

Now, for $p \in [0, A/N]$, which insures that the right-hand side of (A.1) is positive, let $c_t^{n,t}(p)$ be the solution for $c_t^{n,t}$ to Problem 1 when $r_{t+1} = i$ and $\tau_{t+1}^n = \theta^n Npi$, and let $C_t^t(p) = \sum_{n=1}^{N} c_t^{n,t}(p)$. It follows that $C_t^t(p)$ is continuous in p. We need to show that there exists $p \in (0, A/N)$ that satisfies $Np + C_t^t(p) \equiv h(p) = A$. By Assumption 1, we have $h(0) < A$. We also have $h(A/N) > A$. Therefore, there exists $p \in (0, A/N)$ that is a Definition 2 equilibrium. □

Proof of Proposition 2. PROOF. This is another application of the intermediate value theorem. Starting at $t = 2$, this economy is the same as the Proposition 1 equilibrium. Therefore, we let $p_t = p > 0$ for $t \ge 2$ where p is given by Proposition 1. Then, we must have $\sum_{n=1}^{N} \tau_2^n = Npi_2$. Our task is to find p_1 that satisfies the equilibrium conditions. We first construct the lump-sum taxes payable by generation 1 when they are old. We construct such a scheme for each $p_1 > 0$. Let

$$r_2(p_1) = (1+i_2)(p/p_1) - 1, \tag{A.2}$$

the date-1 real interest rate implied by a p_1. Then, we let $\theta_n(p_1) = [a^n + b^n/(1 + r_2(p_1))]/[A + B/(1 + r_2(p_1))]$ and let $\tau_2^n(p_1) = \theta_n(p_1)Npi_2$. Notice that $\theta_n(p_1) \in (0,1)$ and that $\sum_{n=1}^{N} \theta_n(p_1) = 1$. Then the budget set of person n in generation 1 is

$$c_1^{n,1} + \frac{c_2^{n,1}}{1+r_2(p_1)} \le \left(a^n + \frac{b^n}{1+r_2(p_1)}\right)\left[1 - \frac{Npi_2}{A(1+r_2(p_1))+B}\right]. \tag{A.3}$$

Now let $c_2^{n,1}(p_1)$ be the choice of consumption when old and let $C_2^1(p_1) = \sum_{n=1}^N c_2^{n,1}(p_1)$. We want to show that there exists a positive p_1 that satisfies $C_2^1(p_1) = A + B - C_2^2(p)$, where $C_2^2(p) \in (0, A)$ is the consumption when young of generation 2 in the Proposition 1 equilibrium. In other words, we have to show that there exists p_1 that makes consumption when old of generation 1 equals to what it is in the Proposition 1 equilibrium.

Suppose that $i_2 > i$. Let $\bar{p}_1$ be such that $r_2(\bar{p}_1) = i$. Then the budget set at $r_2(p_1) = r_2(\bar{p}_1)$ in (A.3) differs from that in (A.1) only through an income effect: there is less wealth in (A.3) than in (A.1). That implies, via our normal-goods assumption, that $C_2^1(\bar{p}_1) < A + B - C_2^2(p)$. As $p_1 \to 0$, $r_2(p_1) \to \infty$. It follows from the normal goods assumption and from (A.3) that as $r_2(p_1) \to \infty$, $C_2^1(p_1) \to \infty$, and therefore, exceeds $A + B - C_2^2(p)$. By the intermediate value theorem, it follows that there exists some $p_1 \in (0, \bar{p}_1)$ that implies $C_2^1(p_1) = A + B - C_2^2(p)$. Notice that we also conclude in this case that there is an equilibrium in which $r_2(p_1) > i$. That is, a one-date increase in the nominal rate is accompanied by a one-date increase in the real rate in this equilibrium.

Now, suppose that $i_2 < i$. Again, let $\bar{p}_1$ be such that $r_2(\bar{p}_1) = i$, which implies $\bar{p}_1 < p$. Then the budget set at $r_2(p_1) = r_2(\bar{p}_1)$ in (A.3) differs from that in (A.1) only through an income effect: there is more wealth in (A.3) than in (A.1). That implies via our normal-goods assumption that $C_2^1(\bar{p}_1) > A + B - C_2^2(p)$. As $p_1 \to \infty$, $r_2(p_1) \to 0$. Notice that as $r_2(p_1) \to 0$, the budget set, (A.3), approaches the no-tax budget set and implies $C_2^1(p_1) < B$, and therefore, $C_2^1(p_1) < A + B - C_2^2(p)$. By the intermediate value theorem, it follows that there exists some $p_1 > \bar{p}_1$ that implies $C_2^1(p_1) = A + B - C_2^2(p)$. Notice that we also conclude in this case that there is an equilibrium in which $r_2(p_1) < i$. That is, a one-date decrease in the nominal rate is accompanied by a one-date decrease in the real rate in this equilibrium. □

Proof of Proposition 3.

Proof of Proposition 3.1. Proof. Consistent with (6), we define the "stacked" first-order difference equation by

$$\begin{bmatrix} k_{t+2} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} w_{t+1} - zf'(k_{t+1})(w_t - k_{t+1})/(1+i) \\ k_{t+1} \end{bmatrix}. \tag{A.4}$$

Let $\bar{k}$ be the steady state for capital. Let the Jacobian matrix of (A.4) be A,

$$A = \begin{bmatrix} -\bar{k}zf''(\bar{k}) - \frac{zf''(\bar{k})\left(zf(\bar{k}) - \bar{k}zf'(\bar{k}) - \bar{k}\right) - zf'(\bar{k})}{i+1} & \frac{\bar{k}z^2 f'(\bar{k}) f''(\bar{k})}{i+1} \\ 1 & 0 \end{bmatrix}.$$

We first verify that $A - I$ is nonsingular ($|A - I| \neq 0$) to ensure that $\bar{k}$ exists.

$$\begin{aligned} |A - I| &= 1 + \bar{k}zf''(\bar{k}) + \frac{zf''(\bar{k})\left(zf(\bar{k}) - 2\bar{k}zf'(\bar{k}) - \bar{k}\right) - zf'(\bar{k})}{i+1} \\ &= \frac{f''(\bar{k})\left(zf(\bar{k}) - \bar{k}\right)}{f'(\bar{k})} - \bar{k}zf''(\bar{k}) \\ &< 0, \end{aligned}$$

where the second equality uses $i + 1 = zf'(\bar{k})$ and the inequality follows from Assumption 2.

The characteristic equation $|A - \lambda I| = 0$ is equivalent to:

$$\lambda\left[\lambda + \bar{k}zf''(\bar{k}) + \frac{zf''(\bar{k})\left(zf(\bar{k}) - \bar{k}zf'(\bar{k}) - \bar{k}\right) - zf'(\bar{k})}{i+1}\right] - \frac{\bar{k}z^2 f'(\bar{k}) f''(\bar{k})}{i+1} = 0.$$

By using $i + 1 = zf'(\bar{k})$ and Vieta's formulas, we have

$$\lambda_1 + \lambda_2 = -\bar{k}zf''(\bar{k}) - \frac{f''(\bar{k})\big(zf(\bar{k}) - \bar{k}\big)}{f'(\bar{k})} + \bar{k}zf''(\bar{k}) + 1$$
$$= 1 - \frac{f''(\bar{k})\big(zf(\bar{k}) - \bar{k}\big)}{f'(\bar{k})} > 0,$$

and

$$\lambda_1 \cdot \lambda_2 = -\bar{k}zf''(\bar{k}) > 0.$$

It follows that $\lambda_1 > 0$ and $\lambda_2 > 0$. If the eigenvalues satisfy $0 < \lambda_1 < 1 < \lambda_2$ and $tr(A) > 1 + det(A)$, then we have saddle-path stability. By using the result $| A - I |< 0$, we get $0 < \lambda_1 < 1 < \lambda_2$. And to prove $tr(A) > 1 + det(A)$, we have

$$tr(A) - (1 + det(A)) = 1 - \frac{f''(\bar{k})\big(zf(\bar{k}) - \bar{k}\big)}{f'(\bar{k})} - (1 - \bar{k}zf''(\bar{k}))$$
$$= -\frac{f''(\bar{k})\big(zf(\bar{k}) - \bar{k}zf'(\bar{k}) - \bar{k}\big)}{f'(\bar{k})} > 0.$$

The first equality uses $tr(A) = \lambda_1 + \lambda_2$ and $det(A) = \lambda_1 \cdot \lambda_2$.

Therefore, the system is saddle path stable. □

Proof of Proposition 3.2. Proof. For $t = 2$ and given k_2, according to (4), i_2 satisfies

$$zf'(k_2) = \frac{p_2(1 + i_2)}{p_1}$$
$$= \frac{p_2(1 + i_2)}{w_1 - k_2}$$
$$= \frac{p_2(1 + i_2)}{\bar{w} - k_2}$$
$$= \frac{(zf(k_2) - k_2 zf'(k_2) - k_3)(1 + i_2)}{\bar{w} - k_2},$$

where the second equality uses (3), the third one uses $w_1 = \bar{w}$, and the last one applies (2) and (3). Because k_3 is on the saddle path, if the initial point is (k_2, k_3) and $i_t = i$ for all $t > 2$, then the system converges to the steady state.

Therefore,

$$1 + i_2 = \frac{zf'(k_2)(\bar{w} - k_2)}{zf(k_2) - k_2 zf'(k_2) - k_3}. \tag{A.5}$$

Without loss of generality, let $k_2 > \bar{k}$. The derivative of i_2 with respect to k_2 is,

$$\frac{di_2}{dk_2}$$
$$= \frac{[zf''(k_2)(\bar{w}-k_2) - zf'(k_2)]\cdot\Delta - [zf'(k_2)(\bar{w}-k_2)]\Big(-zf''(k_2)k_2 - \frac{\partial k_3}{\partial k_2}\Big)}{\Delta^2}$$
$$= \frac{\Lambda - zf'(k_2)\cdot\Delta + [zf'(k_2)(\bar{w}-k_2)]\frac{\partial k_3}{\partial k_2}}{\Delta^2}$$
$$= \frac{\Lambda - zf'(k_2)\cdot\Delta + [zf'(k_2)(\bar{w}-k_2)]\lambda_1}{\Delta^2} + o(k_2 - \bar{k})$$

$$< \frac{\Lambda - zf'(k_2)\cdot\Delta + zf'(k_2)(\bar{w}-k_2)}{\Delta^2} + o(k_2 - \bar{k})$$
$$= \frac{\Lambda - zf'(k_2)(\Delta - \bar{w} + k_2)}{\Delta^2} + o(k_2 - \bar{k})$$
$$< 0,$$

where $\Delta = zf(k_2) - k_2 zf'(k_2) - k_3$ and $\Lambda = zf''(k_2)(\bar{w} - k_2)(zf(k_2) - k_3)$. The first equality uses (A.5) and the second one is a rearrangement. The third one uses $\partial k_3/\partial k_2 = \lambda_1$, which follows from the fact that the saddle path implies $k_3 = \overline{k} + \lambda_1(k_2 - \overline{k}) + o(k_2 - \overline{k})$ when k_2 is close to $\overline{k}$. The fourth inequality uses $0 < \lambda_1 < 1$, the fifth equality is a rearrangement, and the last inequality uses $zf(k_2) - k_2 zf'(k_2) > \bar{w}$ and $k_2 > k_3$.

Therefore, i_2 is strictly decreasing in k_2 in the neighborhood of $\bar{k}$. □

Proof of Proposition 4. PROOF. Denote the largest and the smallest element of $\mathbb{Z}$ as z^H and z^L. Let $\mathbb{K} = [\underline{k}, \overline{k}] \subset \mathbb{R}$ be the support of the state variable k_t, where $\underline{k}$ is the smallest nonzero solution to the equation $k = (1-\gamma)z^L(f(k) - f'(k)k)$ and $\overline{k}$ is the largest nonzero solution to the equation $k = (1-\gamma)z^H(f(k) - f'(k)k)$. Assumption 3 insures that $\mathbb{K}$ is well defined.

Denote the state space as $\mathbb{S} = \mathbb{Z} \times \mathbb{K}$. Let $\mathcal{S}$ be the usual σ-algebra defined on $\mathbb{S}$. A transition function $P : \mathbb{S} \times \mathcal{S} \mapsto [0, 1]$ is given by

$$\text{(A.6)} \qquad \forall s = (z_t, k_t) \in \mathbb{S}, \quad P(s, \mathbf{ds}) = Pr(z_{t+1}|z_t) \cdot \mathbb{I}(k_t \in h^{-1}(z_t, \mathbf{dk})),$$

where $\mathbf{ds} = z_{t+1} \times \mathbf{dk} \in \mathcal{S}$, $\mathbb{I}$ is the indicator function, and the mapping $h^{-1} : \mathcal{K} \mapsto \mathbb{K}$ is defined by $h^{-1}(z, K) = \{k \in \mathbb{K} | (1-\gamma)z(f(k) - f'(k)k) \in K\}$ for all measurable sets $K \in \mathcal{K}$. We also define the Markov operator T associated with the transition function P. Let $B(\mathbb{S}, \mathcal{S})$ be the space of all the bounded and measurable functions in the measurable space $(\mathbb{S}, \mathcal{S})$. Then $T : B(\mathbb{S}, \mathcal{S}) \mapsto B(\mathbb{S}, \mathcal{S})$ is

$$\text{(A.7)} \qquad Tq(s) = \int q(u)P(s, \mathbf{du}) = \sum_{z_{t+1}\in\mathbb{Z}} Pr(z_{t+1}|z_t)q(z_{t+1}, k_{t+1}),$$

where $q \in B(\mathbb{S}, \mathcal{S})$, $\mathbf{du} \in \mathcal{S}$, and $k_{t+1} = (1-\gamma)z_t(f(k_t) - f'(k_t)k_t)$.

Because $\mathbb{S}$ is a compact, complete, and separable metric space, to guarantee the existence of an ergodic distribution, it is sufficient to prove that P satisfies the Feller property. It follows immediately from (A.7) that T maps bounded continuous functions to bounded continuous functions. This proves existence.

To obtain uniqueness, we apply Theorem 12.12 in Stokey et al. (1989). We verify that P is increasing and that P satisfies the Mixing condition (Assumption 12.1 in Stokey et al. 1989). From Equation (A.7), P is increasing. For mixing, let $k^* = \frac{\overline{k}+\underline{k}}{2}$. Let $\{k_t^H\}_{t=1}^{\infty}$ be the sequence generated by the difference equation $k_{t+1} = (1-\gamma)z^H(f(k_t) - f'(k_t)k_t)$ with initial condition $k_0 = \underline{k}$. Similarly, let $\{k_t^L\}_{t=1}^{\infty}$ be that generated by $k_{t+1} = (1-\gamma)z^L(f(k_t) - f'(k_t)k_t)$ with initial condition $k_0 = \overline{k}$. Because of condition (2) in Assumption 3, $\{k_t^H\}_{t=1}^{\infty}$ and $\{k_t^L\}_{t=1}^{\infty}$ converge to $\bar{k}$ and $\underline{k}$, respectively. Therefore, there exists $N > 0$ such that $k_N^H > k^*$ and $k_N^L < k^*$. For any initial state z_0, let

$$\epsilon = \min\{\Pr(z_t = z^H \text{ for } t = 1, 2, \ldots, N|z_0), \Pr(z_t = z^L \text{ for } t = 1, 2, \ldots, N|z_0)\}.$$

By construction, it follows that the N-step transition function $P^N : \mathbb{S} \times \mathcal{S} \mapsto [0, 1]$ satisfies

$$\text{(A.8)} \qquad P^N\Big((z_0, \underline{k}), \mathbb{Z} \times [k^*, \overline{k}]\Big) \geq \epsilon,$$

and

$$P^N\Big((z_0, \overline{k}), \mathbb{Z} \times [\underline{k}, k^*]\Big) \geq \epsilon. \tag{A.9}$$

This verifies the mixing condition and completes the proof of uniqueness. □

Proof of Proposition 5. PROOF. Setting $\dot{C}(t) = \dot{K}(t) = \dot{p}(t) = 0$, the steady-state equilibrium (C^*, K^*, p^*) is characterized by the solution to the following three equations:

$$C^* = \frac{\rho(\rho+\theta)}{i-\theta}(K^* + p^*), \tag{A.10}$$

$$zf'(K^*) = i + \delta, \tag{A.11}$$

and

$$p^* = \frac{(i-\theta)(zf(K^*) - \delta K^*)}{\rho(\rho+\theta)} - K^*. \tag{A.12}$$

Equations (A.10) and (A.11) come from (14) and (16), respectively. And (A.12) is from (A.10) after replacing C^* by the steady-state expression for C from (15). We first argue that existence of a steady state implies $i \in (\theta, \theta+\rho]$. Because utility is in log form, if the steady state exists, then $C^* > 0$, $K^* > 0$, and $p^* > 0$. From (A.10), it is immediate that $i > \theta$ is necessary. Now we prove that $i \leq \theta + \rho$ is also necessary. Suppose that $i > \theta + \rho$. Then, by (A.10), $C^* < (\rho+\theta)(K^* + p^*)$. But, by (A.12), $(\rho+\theta)(K^*+p^*) = (i-\theta)(zf(K^*) - \delta K^*)/\rho > zf(K^*) - \delta K^* = C^*$, where the last equality comes from the aggregate budget constraint at the steady state. This is a contradiction.

For a given i, (A.10)–(A.12) can be solved by one equation at a time. From (A.11), we can solve for K^* as a function of i, denoted as $\kappa(i)$. Substituting $\kappa(i)$ into (A.12), the existence of $p^* > 0$ is equivalent to

$$\varphi(i) \equiv \frac{(i-\theta)(zf(\kappa(i)) - \delta\kappa(i))}{\rho(\rho+\theta)} - \kappa(i) > 0.$$

Taking the derivative of $\varphi(i)$ with respect to i, we obtain

$$\varphi'(i) = \kappa(i)\left[\frac{1}{\rho(\rho+\theta)}\left(\frac{zf(\kappa(i))}{\kappa(i)} - \delta\right) + \frac{\kappa'(i)}{\kappa(i)}\frac{(i+\rho)(i-\rho-\theta)}{\rho(\rho+\theta)}\right].$$

Because f is strictly concave, $zf(\kappa(i))/\kappa(i) - \delta > zf'(\kappa(i)) - \delta = i > 0$, which, by the implicit function theorem, implies that $\kappa'(i) = 1/(zf''(\kappa(i))) < 0$. That and $i \leq \rho + \theta$ imply that $\varphi'(i) > 0$.

Moreover, we have $\varphi(\theta) = -\kappa(\theta) < 0$ and

$$\begin{aligned}
\varphi(\theta+\rho) \quad &= \tfrac{zf(\kappa(\rho+\theta)) - \delta\kappa(\rho+\theta)}{\rho+\theta} - \kappa(\rho+\theta) \\
&= \kappa(\rho+\theta)\Big\{\Big[\tfrac{zf(\kappa(\rho+\theta))}{\kappa(\rho+\theta)} - \delta\Big]\tfrac{1}{\rho+\theta} - 1\Big\} \\
&> \kappa(\rho+\theta)\Big(\tfrac{zf'(\kappa(\rho+\theta)) - \delta}{\rho+\theta} - 1\Big) \\
&= \kappa(\rho+\theta)\Big(\tfrac{\rho+\theta+\delta-\delta}{\rho+\theta} - 1\Big) = 0,
\end{aligned}$$

which gives us $\varphi(\theta+\rho) > 0$. As a consequence, from the intermediate value theorem, there exists a unique $i^* \in (\theta, \theta+\rho)$ such that $\varphi(i^*) = 0$ and $\varphi(i) > 0$ for all $i \in (i^*, \rho+\theta]$. This proves the existence and uniqueness of p^* under the given i. Finally, the equilibrium consumption C^* is given by (A.10). □

Proof of Proposition 6.

PROOF OF PROPOSITION 6.1. PROOF. It is obvious that one of the eigenvalues of J is $-\eta < 0$. We focus on the remaining three. The characteristic polynomial of $J_{3\times 3}$ (the matrix generated by deleting the fourth row and column of J) is

$$\begin{aligned}\lambda[\lambda-(zf'(K^*)-\delta-\theta)][\lambda-(zf'(K^*)-\delta)]\\ -\rho(\rho+\theta)p^*zf''(K^*)+\lambda[C^*zf''(K^*)-\rho(\rho+\theta)]=0,\end{aligned} \tag{A.13}$$

or

$$\begin{aligned}&\lambda^3-(2zf'(K^*)-2\delta-\theta)\lambda^2+\\ &[2zf'(K^*)-2\delta-\theta+C^*zf''(K^*)-\rho(\rho+\theta)]\lambda-\rho(\rho+\theta)p^*zf''=0.\end{aligned} \tag{A.14}$$

Let these three eigenvalues be λ_i for $i = 1, 2, 3$. From the property of cubic equations, it follows that

$$\lambda_1\lambda_2\lambda_3=\rho(\rho+\theta)p^*zf''(K^*)<0, \tag{A.15}$$

and

$$\sum_{i=1}^{3}\lambda_i=-\theta+2(zf'(K^*)-\delta))>0. \tag{A.16}$$

For a cubic equation, either all three eigenvalues are real, or there are two complex eigenvalues that are conjugate with each other and one real eigenvalue. If all eigenvalues are real, then (A.15) implies that either there is one negative eigenvalue or all three eigenvalues are negative, which is ruled out by (A.16). If there are two conjugate complex eigenvalues, then the real eigenvalue must be negative and both of the complex eigenvalues have positive real parts. In either case—either all real roots or one real root—we conclude that J has two roots with strictly negative real parts and two roots with strictly positive real parts. □

PROOF OF PROPOSITION 6.2. PROOF. One negative eigenvalue of J is $-\eta$ and let the other be denoted as $-\gamma$. Let us first focus on the case that $\gamma \neq \eta$. The general solution to the linearized dynamic system is

$$\begin{bmatrix}C(t)-C^*\\ K(t)-K^*\\ p(t)-p^*\\ e(t)\end{bmatrix}=c_1e^{-\gamma t}V_1+c_2e^{-\eta t}V_2,$$

where $V_1 \equiv [V_{1,1}, V_{1,2}, V_{1,3}, V_{1,4}]'$ and $V_2 \equiv [V_{2,1}, V_{2,2}, V_{2,3}, V_{2,4}]'$ are eigenvectors corresponding to $-\gamma$ and $-\eta$, respectively; c_1 and c_2 are two constant terms that are derived from the initial values, $e(0) = e_0$ and $K(0) = K^*$.

First of all, since V_2 is the eigenvector corresponding to $-\eta$, we obtain

$$J_\eta V_2=\mathbf{0},$$

where

$$J_\eta = \begin{bmatrix} zf'(K^*) - \delta - \theta + \eta & C^* zf''(K^*) - \rho(\rho+\theta) & -\rho(\rho+\theta) & 0 \\ -1 & zf'(K^*) - \delta + \eta & 0 & 0 \\ 0 & zf''(K^*)p^* & \eta & -p^* \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $V_{2,4} = 0$, we can easily show $V_2 = \mathbf{0}$, which contradicts that V_2 is an eigenvector. Normalizing $V_{2,4}$ to be 1, we can solve the value of $V_{2,2}$,

$$F(\eta) \cdot V_{2,2} = \rho(\rho+\theta)\frac{p^*}{\eta} > 0, \tag{A.17}$$

where $F(\cdot)$ is a function $\mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = (zf'(K^*) - \delta - \theta + x)(zf'(K^*) - \delta + x) + [C^* zf''(K^*) - \rho(\rho+\theta)] \\ +\rho(\rho+\theta)\left(\tfrac{1}{x} zf''(K^*)p^*\right).$$

Notice that we have $F(\gamma) = 0$ from Equation (A.13). In addition,

$$F'(x) = (zf'(K^*) - \delta - \theta + x) + (zf'(K^*) - \delta + x) \\ -\rho(\rho+\theta)\left(\tfrac{1}{x^2} zf''(K^*)p^*\right) > 0 \quad \text{for all} \quad x > 0.$$

Hence, if $\eta > \gamma$, we have

$$F(\eta) > F(\gamma) = 0.$$

Combining with (A.17), we have $V_{2,2} > 0$. On the other hand, if $\eta < \gamma$, $F(\eta) < F(\gamma) = 0$, we get $V_{2,2} < 0$ from (A.17). Therefore, we have

$$(\gamma - \eta) \cdot V_{2,2} < 0. \tag{A.18}$$

Since V_1 is the eigenvector corresponding to $-\gamma$, we get

$$J_\gamma V_1 = \mathbf{0}, \tag{A.19}$$

where

$$J_\gamma = \begin{bmatrix} zf'(K^*) - \delta - \theta + \gamma & C^* zf''(K^*) - \rho(\rho+\theta) & -\rho(\rho+\theta) & 0 \\ -1 & zf'(K^*) - \delta + \gamma & 0 & 0 \\ 0 & zf''(K^*)p^* & \gamma & -p^* \\ 0 & 0 & 0 & \gamma - \eta \end{bmatrix}.$$

We immediately have $V_{1,4} = 0$ since $\gamma \neq \eta$. Similarly, we can easily prove that $V_{1,2} \neq 0$; otherwise, $V_1 = \mathbf{0}$. Therefore, we can normalize $V_{1,2}$ to be 1.

Now let us calculate the vales of c_1 and c_2. Given $V_{1,4} = 0$, we have $e(0) = e_0 = c_2 e^{-\eta \times 0} \times V_{2,4}$. Given $V_{2,4} = 1$, we have $c_2 = e_0$. Given $V_{1,2} = 1$, we have $K(0) - K^* = c_1 e^{-\gamma \times 0} + e_0 e^{-\eta \times 0} V_{2,2}$. This implies $c_1 = -e_0 V_{2,2}$ if $K(0) = K^*$. Therefore,

$$\dot{K}(t) = \left(\gamma e^{-\gamma t} - \eta e^{-\eta t}\right) e_0 \cdot V_{2,2}.$$

Driving t to 0^+ and using Equation (A.18), we obtain

$$\lim_{t \to 0^+} \frac{\partial \dot{K}(t)}{\partial e_0} = (\gamma - \eta) \cdot V_{2,2} < 0.$$

Now let us consider the case that $\gamma = \eta$. The general solution to the linearized dynamic system is

$$\begin{bmatrix} C(t) - C^* \\ K(t) - K^* \\ p(t) - p^* \\ e(t) \end{bmatrix} = (c_1 + c_2 \cdot t)e^{-\gamma t}V_1 + c_2 \cdot e^{-\gamma t}\tilde{V}_2,$$

where $V_1 \equiv [V_{1,1}, V_{1,2}, V_{1,3}, V_{1,4}]'$ is the eigenvector corresponding to $-\eta$; and $\tilde{V}_2 \equiv [\tilde{V}_{2,1}, \tilde{V}_{2,2}, \tilde{V}_{2,3}, \tilde{V}_{2,4}]'$ is obtained by solving the nonhomogeneous problem $J\tilde{V}_2 + \eta\tilde{V}_2 = V_1$ for $\tilde{V}_2$. c_1 and c_2 are two constant terms that are derived from the initial values, $e(0) = e_0$ and $K(0) = K^*$.

From the initial condition, we obtain

$$e(0) = e_0 = c_1 \cdot V_{1,4} + c_2 \cdot \tilde{V}_{2,4},$$

and

$$K(0) - K^* = 0 = c_1 V_{1,2} + c_2 \cdot \tilde{V}_{2,2}.$$

As in the case that $\gamma \neq \eta$, we can normalize $V_{1,2} = 1$. By solving Equation (A.19) and using Equation (A.13), we get the eigenvector

$$V_1 = \begin{bmatrix} zf'(K^*) - \delta + \eta \\ 1 \\ -\frac{1}{\eta} zf''(K^*)p^* \\ 0 \end{bmatrix}. \tag{A.20}$$

Given $V_{1,4} = 0$, it then follows that $c_2 = e_0/\tilde{V}_{2,4}$ and $c_1 = -e_0 \cdot \tilde{V}_{2,2}/\tilde{V}_{2,4}$. Substituting back, we get

$$K(t) - K^* = \left(-e_0 \cdot \frac{\tilde{V}_{2,2}}{\tilde{V}_{2,4}} + \frac{e_0}{\tilde{V}_{2,4}} \cdot t\right)e^{-\gamma t} + \frac{e_0}{\tilde{V}_{2,4}} \cdot \tilde{V}_{2,2} \cdot e^{-\gamma t} = \frac{e_0}{\tilde{V}_{2,4}} \cdot t \cdot e^{-\gamma t}.$$

Taking derivation with respect to t and letting $t \to 0^+$, we have

$$\lim_{t \to 0^+} \dot{K}(t) = \frac{e_0}{\tilde{V}_{2,4}}.$$

To get the sign of $\tilde{V}_{2,4}$, we simplify the system of $J\tilde{V}_2 + \eta\tilde{V}_2 = V_1$ by removing $\tilde{V}_{2,1}$, $\tilde{V}_{2,2}$, and $\tilde{V}_{2,3}$:

$$\begin{aligned} &-\rho(\rho+\theta)\frac{1}{\eta}p^*\tilde{V}_{2,4} = \\ &V_{1,1} + (zf'(K^*) - \delta - \theta + \eta)V_{1,2} + \rho(\rho+\theta)\frac{1}{\eta}V_{1,3}. \end{aligned} \tag{A.21}$$

Plugging the value of each element of V_1 (see (A.20)) into (A.21), we obtain

$$\begin{aligned} \tilde{V}_{2,4} &= \frac{(zf'(K^*)-\delta+\eta)+(zf'(K^*)-\delta-\theta+\eta)-\rho(\rho+\theta)\frac{1}{\eta}\left(\frac{1}{\eta}zf''(K^*)p^*\right)}{[-\rho(\rho+\theta)]\frac{1}{\eta}p^*} \\ &< 0. \end{aligned}$$

This completes the proof that $\lim_{t \to 0^+} \frac{\partial \dot{K}(t)}{\partial e_0} < 0$. □

SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Data S1
Online Appendix Figure 1: Impulse Response Functions of k_t and p_t
Table 1: Parameters
Online Appendix Figure 2: Impulse Response Functions

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